ME 305 Fluid Mechanics I

Part 5

Bernoulli Equation

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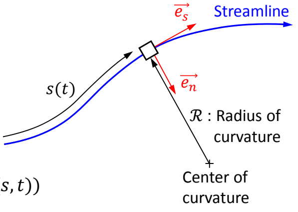
Bernoulli Equation (BE)

- BE is a simple and easy to use relation between the following three variables in a moving fluid
  - pressure
  - velocity
  - elevation
- It can be thought of a limited version of the 1st law of thermodynamics.
- It can also be derived by simplifying Newton's 2nd law of motion written for a fluid particle moving along a streamline in an inviscid fluid.
- **Warning:** BE is the most used and the most abused equation in fluid mechanics.
  So be careful !!!

Movie: Streamlines

Streamline coordinates

- Consider a fluid particle moving along a streamline in a planar flow.
- Its current position is given by $s(t)$. Its speed is $V = ds/dt$.
- $\overrightarrow{e_s}$ is the direction along the streamline.
- If the streamline is curved, $\overrightarrow{e_n}$ points towards the center of curvature.
- Acceleration components are



$$a_s = \frac{dV}{dt} = \frac{\partial V}{\partial t} + V\frac{\partial V}{\partial s} \quad \text{(Note that } V = V(s,t)\text{)}$$

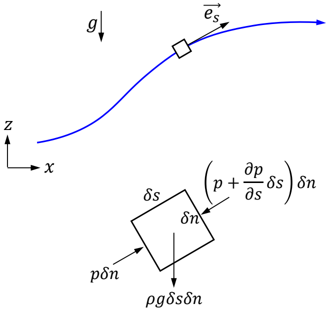
$$a_n = \frac{V^2}{\mathcal{R}} \quad \text{(= zero for a straight streamline)}$$

- Pressure, viscous and body forces acting on the particle create these accelerations.
- BE is derived by considering the s component of this force balance in the absence of viscous forces.

Derivation of the BE

- Main assumption behind the BE is the omission of viscous forces (inviscid flow).

Exercise : A fluid particle moves along a streamline for the following 2D, steady flow in the xz plane. Derive the steady version of the BE by considering the pressure and body forces in the streamline direction.



Newton's 2nd law for inviscid flow

Net pressure force on the particle
+
Net body force on the particle
=
(Particle's mass) x (Particle's acceleration)

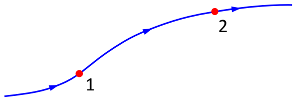
The Most Commonly Used Form of the BE

- The following most common form of the BE is valid for steady, incompressible, inviscid flows.

$$\boxed{p + \frac{\rho V^2}{2} + \rho g z = \text{constant along a streamline}}$$

- Using two points on a streamline

$$\boxed{p_1 + \frac{\rho V_1^2}{2} + \rho g z_1 = p_2 + \frac{\rho V_2^2}{2} + \rho g z_2}$$



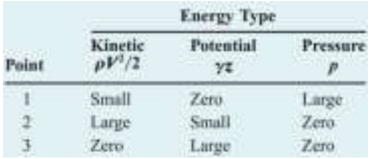
- BE can also be understood as "Work done on a fluid particle by pressure and gravity forces is equal to the change in its kinetic energy".

Exercise : Compare the above BE with the energy conservation equation written for a uniform, steady flow in a single inlet single exit CV.

- Note that more general forms of the BE also exist for compressible, unsteady flows.

The Most Commonly Used Form of the BE (cont'd)

- BE can be seen as a balance of kinetic, potential and pressure energies.
- Consider the flow of water from the syringe. The force applied to the plunger will produce a pressure greater than atmospheric pressure at point 1. The water flows from the needle (point 2) with relatively high velocity and rises up to point 3 at the top of its trajectory (Reference: Munson's book).



Point	Energy Type		
	Kinetic $\rho V^2/2$	Potential γz	Pressure p
1	Small	Zero	Large
2	Large	Small	Zero
3	Zero	Large	Zero

- Due to the friction effects (viscous forces) the water will not go up as much as predicted by the BE.
- Such effects arise especially at the narrow needle exit and between the water jet and surrounding air stream.

BE in "Head" Form

- Divide all the terms of the BE by ρg

$$\underbrace{\frac{p}{\rho g} + \frac{V^2}{2g} + z}_{\text{Total head } (h_T)} = \text{constant along a streamline}$$

Pressure head

Velocity head

Elevation head

Total head (h_T)

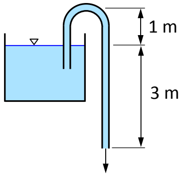
BE says that total head is constant along a streamline.

- In this form all the terms have the units of length and they are called heads.
- Elevation head : related to the potential energy of the fluid.
- Pressure head : represents the height of column of the fluid that is needed to produce the pressure p.
- Velocity head : represents the vertical distance needed for the fluid to fall freely (neglecting friction) if it is to reach velocity V from rest.

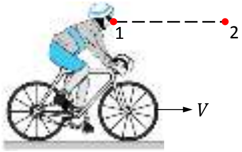
Bernoulli Equation Exercises (cont'd)

Exercise : A tube can be used to discharge water from a reservoir as shown. Determine the speed of the free jet and the minimum absolute pressure of water that occurs at the top of the bend.

This is known as siphoning. It can be used to drain gas from the tank of an automobile. Once you establish the initial flow by sucking gas from the tube, the gas will flow by itself.

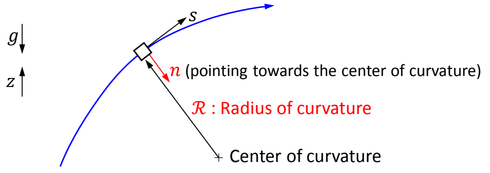


Exercise : Consider the flow of air around a cyclist moving through still air with velocity V. Determine the pressure difference between points 1 and 2.
Hint : Be careful about the unsteadiness of the flow field.



Pressure Variation Normal to the Streamlines

- We can also study the force balance normal to the streamline (in the n direction).



n (pointing towards the center of curvature)

$\mathcal{R}$: Radius of curvature

Center of curvature

- For steady flows we obtain the following equation (see Munson's book for details)

$$-\rho g \frac{dz}{dn} - \frac{\partial p}{\partial n} = \rho \frac{V^2}{\mathcal{R}}$$

($V^2/\mathcal{R}$) is the centrifugal acceleration due to direction change

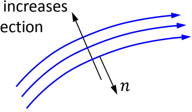
Pressure Variation Normal to the Streamlines (cont'd)

- $\mathcal{R} = \infty$ corresponds to a straight streamline.
- $\mathcal{R} < \infty$ corresponds to a change in the flow direction, i.e. a curved streamline. This is accomplished by the appropriate combination of pressure gradient and fluid weight normal to the streamline.
- If gravitational effects are negligible (gas flows), or if the flow is in a horizontal plane

$$-\frac{\partial p}{\partial n} = \frac{\rho V^2}{\mathcal{R}}$$

- Therefore when streamlines are curved, pressure increases with distance away from the center of curvature.
- The pressure difference is used to balance the centrifugal acceleration associated with the curved streamlines.

Movie
Free vortex

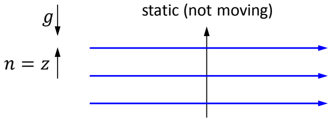


Pressure increases in $-\vec{n}$ direction

Curved streamlines of a flow field

Pressure Variation Normal to the Streamlines (cont'd)

- For straight and parallel streamlines ($\mathcal{R} = \infty$) pressure variation across the streamlines is hydrostatic (as if the fluid is not moving)



Pressure variation normal to the streamlines is as if the fluid is static (not moving)

Straight and parallel streamlines of a flow field ($\mathcal{R} = \infty$)

From Slide 5-9: $-\rho g - \frac{dp}{dz} = 0$ (Fluid statics equation)

- This fact will be used in studying speed measurement with a Pitot tube.

Static, Dynamic, Stagnation & Total Pressures

$$\underbrace{\underbrace{\underset{\text{Static pressure}}{p} + \underset{\text{Dynamic pressure}}{\frac{\rho V^2}{2}}}_{\text{Stagnation pressure}} + \underset{\text{Hydrostatic pressure}}{\rho g z}}_{\text{Total pressure}} = \text{constant along a streamline}$$

BE says that total pressure is constant along a streamline.

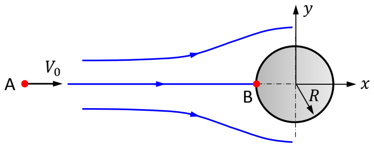
- Static pressure is also known as the thermodynamic pressure. To measure it one could move with the fluid, thus being static relative to the moving fluid. It can also be measured using a piezometer tube (as will be seen later).
- Dynamic pressure represents the rise in pressure as a fluid slows down along a streamline (see the next page).
- At a stagnation point $V = 0$, and stagnation and static pressures are equal.

Static, Dynamic, Stagnation & Total Pressures (cont'd)

Exercise : Consider the inviscid, incompressible, steady flow along the horizontal streamline A-B in front of the sphere. Analytical work yields the following fluid velocity equation along this streamline.

$$V = V_0\left(1 + \frac{R^3}{x^3}\right)$$

Determine the pressure variation along the streamline from point A far away from the sphere ($x_A \rightarrow -\infty$, $V_A = V_0$) to point B on the sphere ($x_B \rightarrow -R$, V_B =0).



Simple Pitot Tube

- Pitot tube is a device used for speed measurement.
- It is a simple tube with a 90 degree bend.
- It measures flow speed using the Bernoulli principle.

Pitot tube on a Formula 1 car

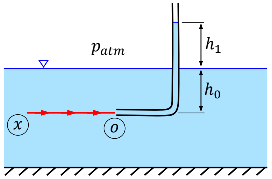
http://www.daylife.com/photo/00sl2OC8dr9rc

Pitot tubes on a passanger aircraft

- Read about the role of Pitot tube malfunctions on plane crashes.

http://www.associatedcontent.com/article/1362487/plane_crashes_and_pitot_tubes.html?cat=15

Simple Pitot Tube (cont'd)



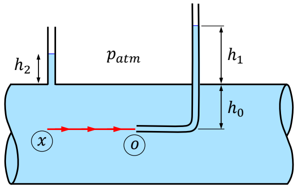
- Fluid flows in an open channel from left to right.
- We want to measure the speed at point x.
- Fluid fills the Pitot tube and rises inside it to a level of h_1 above the free surface.
- The aim of using a Pitot tube is to create a stagnation point at point "o" with zero velocity.

Exercise: Show that the fluid speed at point x is given by

$$\boxed{V_x = \sqrt{2gh_1}}$$

- With a Pitot tube we actually measure the pressure difference between points "x" and "o" and convert this difference to a speed difference using the BE.

Use of Pitot Tube with a Piezometer



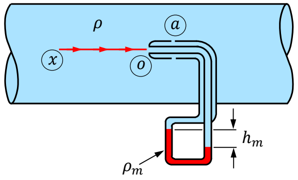
- For the flow in a closed channel or pipe we need to use an additional tube called the piezometer tube (static tube).
- Piezometer is used to measure the static pressure at point x as

$$p_x = p_{atm} + \rho g(h_0 + h_2)$$

- Using the Pitot tube : $p_o = p_{atm} + \rho g(h_0{+}h_1)$
- BE between points x and o gives the unknown speed as : $V_x = \sqrt{2(p_o - p_x)/\rho}$

$$\boxed{V_x = \sqrt{2g(h_1 - h_2)}}$$

Combined Pitot-Static Tube (Prandtl's Tube)

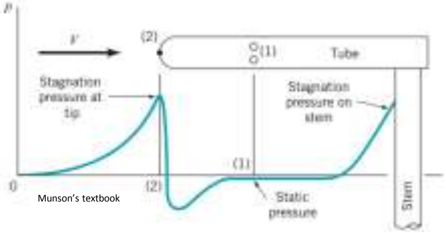


- Instead of measuring static pressure at point x using a piezometer tube, a second tube is used around the Pitot tube.
- Static pressure holes (point a) of the outer tube are located such that they measure correct upstream static pressure, i.e. $p_a = p_x$.
- Two tubes provide the necessary pressure difference measurement using the mercury in it.
- It is possible to use pressure transducers instead of mercury columns to obtain accurate digital readings.
- The required pressure difference is $p_o - p_x = (\rho_m - \rho) g h_m$
- Using this in the BE we get

$$V_x = \sqrt{2(p_o - p_x)/\rho} \quad \rightarrow \quad \boxed{V_x = \sqrt{2 g h_m \left(\frac{\rho_m}{\rho} - 1\right)}}$$

Combined Pitot-Static Tube (cont'd)

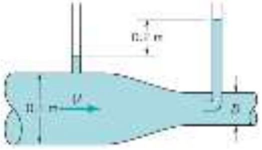
- Typical pressure variation along a combined Pitot-static tube is as shown.
- As seen, the holes are located such that they measure the static pressure ahead of the device.



Munson's textbook

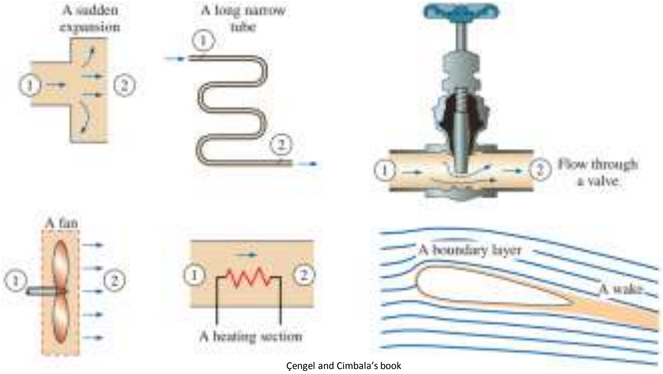
Exercise: Water flows through the pipe contraction shown. For the given 0.2 m difference in the manometer level, determine the flow rate if the small pipe diameter is

a) $D = 0.05$ m, b) $D = 0.03$ m



Be Careful in Using the Bernoulli Equation

- The simplest and the most commonly used BE that we studied in the previous slides may lead to unphysical results for problems similar to the following ones.
- BE will be extended in the next slide to solve some of these problems.



Çengel and Cimbala's book

Extended Bernoulli Equation (EBE)

- It is a modified version of the BE to include effects such as viscous forces, heat transfer and shaft work.
- Remember the energy conservation equation for a single inlet, single exit CV with uniform properties.

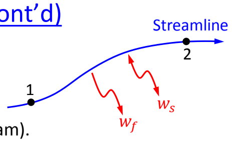
$$q + w_s = \left(u + \frac{p}{\rho} + \frac{V^2}{2} + gz\right)_{\text{exit}} - \left(u + \frac{p}{\rho} + \frac{V^2}{2} + gz\right)_{\text{inlet}}$$

- Arranging this equation we get

$$\underbrace{\left(\frac{p}{\rho} + \frac{V^2}{2} + gz\right)_{\text{inlet}} = \left(\frac{p}{\rho} + \frac{V^2}{2} + gz\right)_{\text{exit}}}_{\text{Original BE}} + \underbrace{(u_{\text{exit}} - u_{\text{inlet}} - q)}_{\text{Frictional work per unit mass } (w_f)} - \underbrace{w_s}_{\text{Shaft work done per unit mass}}$$

Extended Bernoulli Equation (cont'd)

$$\left(\frac{p}{\rho}+\frac{V^2}{2}+gz\right)_1 = \left(\frac{p}{\rho}+\frac{V^2}{2}+gz\right)_2 + w_f - w_s$$



- Flow is from location 1 (upstream) to location 2 (downstream).
- Shaft work (w_s)
  - For a turbine, which converts hydraulic energy into mechanical energy, the work is done by the fluid and w_s is negative.
  - For a pump, which converts mechanical energy into hydraulic energy, the work is done on the fluid and w_s is positive.
- Frictional work (w_f) is the amount of mechanical energy converted into thermal energy due to viscous action.
  - It corresponds to a rise in the internal energy of the fluid (heat up the fluid) or to the heat that is lost to the surroundings.
  - Although possible heat addition to the fluid is also included in this term, it is almost always used to represent a loss (a positive quantity in the above equation).

"Head" Form of the EBE

- Dividing both sides of the EBE by g we get

$$\underbrace{\left(\underbrace{\frac{p}{\rho g}}_{\text{Pressure head}} + \underbrace{\frac{V^2}{2g}}_{\text{Velocity head}} + \underbrace{z}_{\text{Elevation head}}\right)_1}_{\text{Total head at 1 } (h_{T_1})} = \underbrace{\left(\frac{p}{\rho g}+\frac{V^2}{2g}+z\right)_2}_{\text{Total head at 2 } (h_{T_2})} + \underbrace{h_f}_{\text{Friction head}} - \underbrace{h_s}_{\text{Pump or turbine head}}$$

- EBE can simply be written has

$$h_{T_1} = h_{T_2} + h_f - h_s \qquad \text{or} \qquad h_{T_2} = h_{T_1} - h_f + h_s$$

Total head at a downstream location is equal to the total head at an upstream location minus the head loss due to frictional losses plus the head due to shaft work.

Pump and Turbine Head (h_s)

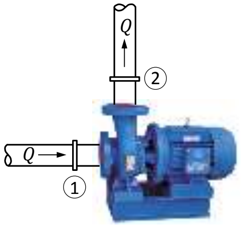
- Pump head h_s is related to the power delivered to the fluid by the pump ($\mathcal{P}_f$) as follows

$$\mathcal{P}_f = \dot{m} w_s = \rho Q w_s \quad \rightarrow \quad \boxed{\mathcal{P}_f = \rho g Q h_s}$$

where Q is the volumetric flow rate that passes through the pump.

- Power delivered to the fluid is related to the power consumed by the pump ($\mathcal{P}_{\text{pump}}$) through the pump efficiency

$$\boxed{\eta_{\text{pump}} = \frac{\mathcal{P}_f}{\mathcal{P}_{\text{pump}}}}$$



Centrifugal pump

- For a turbine, power extracted from the fluid is calculated in a similar way.

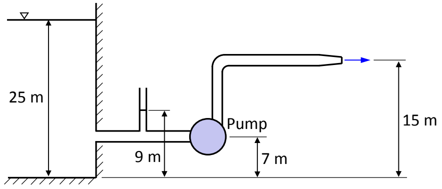
$$\mathcal{P}_f = \rho g Q h_s$$

- Power produced by the turbine ($\mathcal{P}_{\text{turbine}}$) is smaller than the extracted fluid power

$$\boxed{\eta_{\text{turbine}} = \frac{\mathcal{P}_{\text{turbine}}}{\mathcal{P}_f}}$$

Extended BE Exercises

Exercise : The pump shown below pumps water steadily at a volumetric rate of 0.005 m³/s through a constant diameter pipe. At the end of the pipe there is a nozzle with an exit area that is equal to half of the pipe area. Neglecting frictional losses, determine the power that must be supplied to the pump, if it is working with 70 % efficiency.



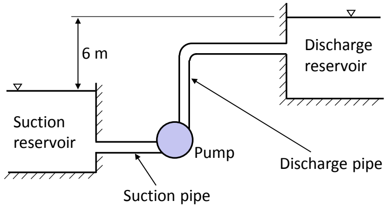
Extended BE Exercises (cont'd)

Exercise : A pump is used to transport water between two large reservoirs. Desired volumetric flow rate through the suction and discharge pipes is 0.016 m³/s. Cross-sectional area of the pipes are 0.004 m². Total frictional head losses between two reservoirs is estimated to be 2 m. Efficiency of the pump is 75 %. Determine

a) the required pump head.

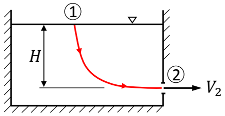
b) the power delivered to the water by the pump.

c) the power required to drive the pump.



Toricelli Equation

- Consider the discharge of a liquid from a large reservoir through an orifice (hole).



- BE between the free surface and the orifice is

$$p_1 + \frac{\rho V_1^2}{2} + \rho g z_1 = p_2 + \frac{\rho V_2^2}{2} + \rho g z_2$$

$$p_1 = p_2 = p_{atm} \quad , \quad z_1 - z_2 = H \quad , \quad V_1 \approx 0$$

$$\boxed{V_2 = \sqrt{2gH}}$$ (Toricelli Equation)

Movie : Toricelli

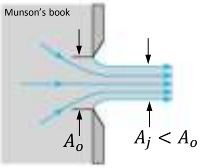
- Discharge through the orifice with an area A_o is

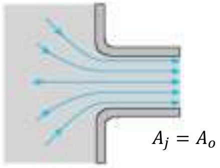
$$Q_{\text{orifice}} = V_2 A_o$$

- This value will be corrected in the following slides.

Vena Contracta and Contraction Coefficient

- Depending on the geometry of the orifice, flow field near the exit may be as follows.



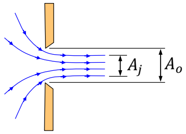


$$C_c = \frac{A_j}{A_o} < 1$$

Contraction coefficient

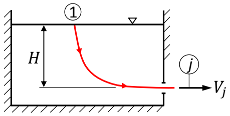
$$C_c = \frac{A_j}{A_o} = 1$$

Vena Contracta and Contraction Coefficient (cont'd)



- *Vena contracta* is the cross section of the jet where the streamlines are straight and parallel.
- This is the section at which pressure is equal to p_{atm}.
- Contraction coefficient : $C_c = A_j / A_o$

- So the correct BE should be written between the free surface and the *vena contracta* section, shown as *j* below.



$$\cancel{p_1} + \overset{\sim 0}{\cancel{\frac{\rho V_1^2}{2}}} + \rho g z_1 = \cancel{p_j} + \frac{\rho V_j^2}{2} + \rho g z_j$$

$$V_j = \sqrt{2gH}$$

- Discharge through the orifice is $Q_{\text{orifice}} = A_j \sqrt{2gH} \quad \rightarrow \quad \boxed{Q_{\text{orifice}} = C_c A_o \sqrt{2gH}}$

Velocity Coefficient and Discharge Coefficient

- The actual discharge would be even less due to viscous effects.
- Velocity coefficient (C_v) corrects this

$$Q_{\text{orifice}} = C_c\ C_v\ A_o\sqrt{2gH}$$

Correction due to geometry

Correction due to viscous effects

- Discharge coefficient (C_d) combines contraction and velocity coefficients

$$C_d = C_c C_v$$

- Therefore discharge through the orifice can be given as

$$\boxed{Q_{\text{orifice}} = C_d A_o\sqrt{2gH}}$$

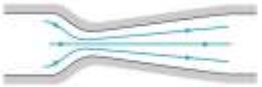
- C_d is determined experimentally for a given orifice geometry and for various flow conditions.

Obstruction Flow Meters

- They are used to measure flow rates through pipes. General idea is
  - to place an obstacle inside the pipe and force the fluid to accelerate and pass from a narrow area.
  - measure the pressure difference between the low-velocity, high-pressure upstream and the high-velocity, low-pressure downstream.
  - use the BE to relate this pressure difference to the flow rate in the pipe.

Orifice meter

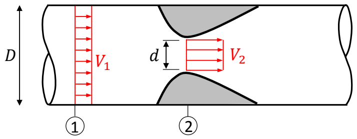
Nozzle flow meter



Venturi meter

Movie : Venturi meter

Venturi Meter



$$A_1 = \frac{\pi D^2}{4} \qquad A_2 = \frac{\pi d^2}{4}$$

- Section 1 is an upstream section with an average velocity of V_1.
- We are interested in measuring V_1.
- Section 2 is the throat of the Venturi. It is also the *vena contracta* due to the smooth profile of the Venturi.
- Difference between p_1 and p_2 is measured by using static holes.

Venturi Meter (cont'd)

- BE between points 1 and 2 located at the centerline

$$p_1 + \frac{\rho V_1^2}{2} = p_2 + \frac{\rho V_2^2}{2}$$

- Continuity equation for a CV between sections 1 and 2

$$A_1 V_1 = A_2 V_2 \quad \rightarrow \quad \frac{\pi D^2}{4} V_1 = \frac{\pi d^2}{4} V_2$$

- Combine these two equations to eliminate V_1 and obtain V_2 as

$$V_2 = \sqrt{\frac{2(p_1 - p_2)}{\rho(1-\beta^4)}} \qquad \text{where} \qquad \beta = \frac{d}{D}$$

- Flow rate through the pipe is given by

$$\boxed{Q = A_2 V_2 = A_2\sqrt{\frac{2(p_1 - p_2)}{\rho(1-\beta^4)}}}$$

Venturi Meter (cont'd)

- This flow rate can be corrected for viscous effects using the discharge coefficient

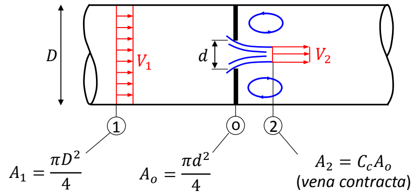
$$\boxed{Q = C_d A_2 \sqrt{\frac{2\,(p_1 - p_2)}{\rho\,(1-\beta^4)}}} \quad , \quad \beta = \frac{d}{D}$$

Experimentally determined and
provided by the manufacturer
(see the Slide 5-35).

- For the nozzle flow meter the same equation can be used.
- Note that for the Venturi meter and the nozzle flow meter, the contraction is smooth and the contraction coefficient is 1 ($C_c = 1$).

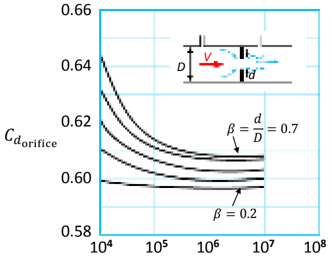
Orifice Meter

- For the orifice meter the expansion is abrupt and C_c is not 1, i.e. *vena contracta* area is smaller than the orifice area.

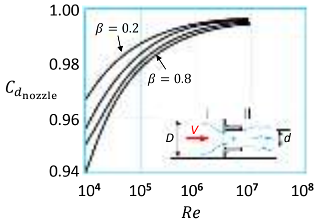


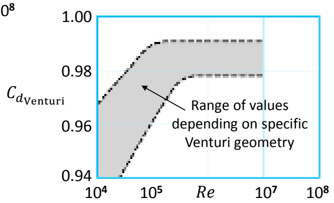
- Section o has the orifice plate with the hole diameter d.
- Section 2 is the *vena contracta* section. p_2 is measured here.
- Following slide 5-32 a new equation can be derived for V_2. The effect of $C_c \neq 1$ will be seen. But in practice the equation of Slide 5-32, derived for Venturi meter, is generally used, with C_d including the effect of C_c too.

C_d Graphs for Obstruction Flow Meters



$$Re = \frac{\rho V D}{\mu}$$

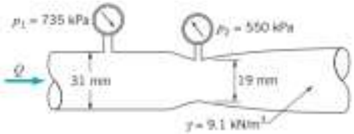




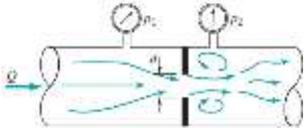
Range of values
depending on specific
Venturi geometry

Obstruction Flow Meter Exercises

Exercise : (Munson's book) a) Determine the flowrate through the Venturi meter shown. b) At what flowrate the cavitation will begin if $p_1 = 376$ kPa and vapor pressure of the flowing fluid is 3.6 kPa.



Exercise : (Munson's book) What diameter orifice hole is needed if under ideal conditions the flowrate through the orifice meter is to be 113 L/min of water with $p_1 - p_2 = 16.34$ kPa ? Pipe diameter is 5 cm and the contraction coefficient is 0.63.



Obstruction Flow Meters (cont'd)

- Comparison of obstruction type flow meters

	Cost	Ease of Installation	Pressure Loss
Orifice meter	Cheap	Difficult	High
Nozzle flow meter	Medium	Difficult	Medium
Venturi meter	High	Difficult	Low

- Other types flow meters
    - Rotameter (youtube.com/watch?v=2dfIWNYJwZM)
    - Thermal flow measurement (youtube.com/watch?v=YfQSf2NBGqc)
    - Vortex type flow meter (youtube.com/watch?v=GmTmDM7jHzA)
    - Ultrasonic flow meter (youtube.com/watch?v=Bx2RnrfLkQg)
    - Coriolis flow measurement (youtube.com/watch?v=XIIViaNITIw)
    - Turbine flow meter
    - Weirs (for open channels)